Int. J. Heat Mass Transfer. Vol. 8, pp. 35–41. Pergamon Press 1965. Printed in Great Britain

CONVECTIVE INSTABILITY OF A HYDROMAGNETIC FLUID WITHIN A RECTANGULAR CAVITY

ULRICH H. KURZWEG

United Aircraft Corporation Research Laboratories, East Hartford, Connecticut, U.S.A.

(*Received* 1 *April* 1964 *and in revised form* 20 *July* 1964)

Abstract—The onset of convection in an electrically conducting fluid confined within an infinite cavity of rectangular cross section is examined for the case when the fluid is subjected to both a vertical temperature gradient and a vertical magnetic field. Characteristic value equations governing fluid stability are derived and a modified Fourier technique is used to solve them. The critical temperature gradient at the onset convection is found to be a function of the Hartmann number and of the cavity aspect ratio.

NOMENCLATURE

$2a$,	width of cavity;
A_{mn}, B_{mn},	expansion coefficients for ψ and θ;
$2b$,	height of cavity;
g,	gravitational acceleration;
h_i,	x_i component of magnetic field;
k,	coefficient of thermal diffusivity;
M,	Hartmann number;
$\bar{p}$,	sum of the hydrodynamic and magnetic pressures;
R,	Rayleigh number;
T,	temperature;
v_i,	x_i component of velocity;
U, V,	expansion functions for ψ;
X, Y,	expansion functions for θ;
x_i	Cartesian co-ordinate.

Greek symbols

α,	coefficient of volume expansion;
β,	conductive temperature gradient;
γ,	cavity aspect ratio ($= a/b$);
Γ, Δ,	definite integrals defined below equation (26);
δ,	wave number of the convection cells;
ζ,	non-dimensional vertical co-ordinate;
η,	non-dimensional horizontal co-ordinate;
θ,	temperature perturbation;
Θ,	vertical component of θ for large γ;
λ,	characteristic values of the even beam functions;
μ,	magnetic permeability;
ν,	kinematic viscosity;
ρ,	density;
σ,	electrical conductivity;
τ,	characteristic values of the odd beam functions;
ϕ,	magnetic stream function;
ψ,	velocity stream function;
Ψ,	vertical component of ψ for large γ.

1. INTRODUCTION

ONE OF the more interesting problems of dissipative hydromagnetics, both because of its relevance to heat transfer in liquid metals under laboratory conditions and because of its possible geophysical importance, is the onset of thermal convection within an electrically conducting fluid subjected to the simultaneous action of an adverse temperature gradient and an externally applied magnetic field. The earliest investigation of this phenomenon was undertaken by Thompson [1] and by Chandrasekhar [2]. Both authors extended the classical Bénard problem of convective instability in a shallow layer of fluid heated from below, to the case where a vertical magnetic field acts on the fluid. The field was found to have a strong stabilizing tendency with the critical Rayleigh number depending upon the magnitude of the Hartmann number. These results were later confirmed experimentally by Nakagawa [3]. Regirer [4] has examined the stability of a conducting fluid between parallel vertical plates for a transverse magnetic field while Sorokin and Sushkin [5] have given a general discussion of hydromagnetic convection within a cavity of arbitrary shape.

It is the purpose of the present paper to examine the stability of a hydromagnetic fluid confined within a cavity of infinite length and rectangular cross section. A similar problem in the absence of a magnetic field has recently been considered by Velte [6]. He examined the stability of laminar flow in a horizontal pipe of square cross section when a temperature gradient acts vertically. It will be shown below that the onset of hydromagnetic convection within the rectangular cavity depends not only on the magnitude of the Hartmann number but also upon the cavity aspect ratio. A comparison with the results of the Bénard problem will indicate that the sidewalls of the cavity have an appreciable stabilizing influence whenever the height–width ratio is greater than unity.

2. FORMULATION OF THE PROBLEM

Consider an incompressible fluid of density ρ, kinematic viscosity ν and electrical conductivity σ confined within a cavity formed by the two vertical planes $x_1 = \pm a$ and the two horizontal planes $x_2 = \pm b$. The lower and the upper walls of the cavity are assumed to be good thermal conductors and maintained at temperature T_1 and T_2 respectively. The sidewalls are perfect insulators and a constant vertical magnetic field H_0 permeates the fluid.

Within the framework of the Boussinesq approximation [7], the equations governing steady-state convection in this cavity are

$$\rho_1 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \rho_1 \nu \nabla^2 v_i + \mu h_j \frac{\partial h_i}{\partial x_j} - \rho(0, g, 0), \quad (1)$$

$$v_j \frac{\partial h_i}{\partial x_j} = h_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\mu\sigma} \nabla^2 h_i, \quad (2)$$

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial h_i}{\partial x_i} = 0, \quad (3)$$

$$\rho = \rho_1[1 - \alpha(T - T_1)], \quad (4)$$

$$v_j \frac{\partial T}{\partial x_j} = k \nabla^2 T, \quad (5)$$

when expressed in Cartesian co-ordinates (x_1, x_2, x_3) and mks units. Here v, h, T and $\bar{p}$ are the velocity, the magnetic field, the temperature and the effective pressure while α is the coefficient of volume expansion and k the thermal diffusivity of the fluid. If we assume that the convection is characterized by infinite horizontal rolls (i.e. no x_3 dependence), then equations (1)–(5) reduce to

$$\frac{\mu}{\rho_1} \frac{\partial(\phi, \nabla^2\phi)}{\partial(x_1,x_2)} - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x_1,x_2)} = \nu \nabla^4\psi + g\alpha \frac{\partial T}{\partial x_1} \quad (6)$$

$$\frac{\partial(\phi, \partial\psi/\partial x_1)}{\partial(x_1, X_2)} - \frac{\partial(\psi, \partial\phi/\partial x_1)}{\partial(x_1,x_2)} = \frac{1}{\mu\sigma} \nabla^2 \frac{\partial\phi}{\partial x_1}, \quad (7)$$

$$\frac{\partial(T,\psi)}{\partial(x_1,x_2)} = k \nabla^2 T, \quad (8)$$

where

$$\frac{\partial(F,G)}{\partial(x_1,x_2)} = \frac{\partial F}{\partial x_1}\left(\frac{\partial G}{\partial x_2}\right) - \left(\frac{\partial G}{\partial x_1}\right)\frac{\partial F}{\partial x_2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

ψ and ϕ are the velocity stream function and the magnetic stream function, respectively. They satisfy the continuity requirements (3) identically.

Since the interest here is not in obtaining information concerning the absolute magnitude of the functions ψ, ϕ, and T, but only in ascertaining the critical temperature gradient at which convection can first occur, it is possible to linearize the above equations. Linearization is accomplished by assuming that the functions are infinitesimal perturbations superimposed upon the stationary conditions $h = H_0$ and $T = T_1 - \beta(b + x_2)$, where $\beta = (T_1 - T_2)/2b$ is the conductive temperature gradient. This procedure leads, after introduction of the dimensionless variables $\eta = x_1/a$ and $\zeta = x_2/b$, to the equations

$$\left[L^2 - (\gamma M)^2 \frac{\partial^2}{\partial \zeta^2}\right] \psi = R \frac{\partial \theta}{\partial \eta}, \quad (9)$$

$$L\theta = \frac{\partial\psi}{\partial\eta}, \quad (10)$$

where

$$L = \frac{\partial^2}{\partial\eta^2} + \gamma^2 \frac{\partial^2}{\partial\zeta^2},$$

$R = g\alpha\beta a^4/k\nu$ = Rayleigh number.
$M = \mu H_0 a(\sigma/\rho\nu)^{1/2}$ = Hartmann number,
$\gamma = a/b$ = cavity aspect ratio,

and θ the perturbation of the conductive temperature profile T. The boundary conditions needed to make these equations determinate follow directly from the non-slip requirement on the velocity at the boundary and from the assumed thermal properties of the walls. The conditions are

$$\left.\begin{aligned} \psi = \frac{\partial\psi}{\partial\eta} = \frac{\partial\theta}{\partial\eta} = 0 \quad \text{at } \eta = \pm 1, \\ \psi = \frac{\partial\psi}{\partial\eta} = \theta = 0 \quad \text{at } \zeta = \pm 1. \end{aligned}\right\} \quad (11)$$

3. SOLUTION FOR LARGE ASPECT RATIO

Before proceeding with a numerical treatment of equations (9) and (10) for arbitrary γ, it is of interest to briefly consider two limiting cases for which simple analytic solutions can be found. The first of these limiting cases corresponds to a cavity of large aspect ratio ($\gamma \to \infty$). Under this circumstance the sidewalls have little influence on the stability of the flow, so that one can assume that the dependent variables have the periodic form

$$\psi = \Psi(\zeta) \exp i\delta\eta, \; \theta = \Theta(\zeta) \exp i\delta\eta, \quad (12)$$

where δ is the dimensionless wave number. Substituting (12) into equations (9) and (10) one obtains

$$[(\gamma^2 D^2 - \delta^2)^2 - (\gamma M)^2 D^2]\Psi = i\delta R\Theta, \quad (13)$$

$$(\gamma^2 D^2 - \delta^2)\Theta = i\delta\Psi, \quad (14)$$

where $D = \mathrm{d}/\mathrm{d}\zeta$. These equations can be combined to yield

$$[D^2 - (\delta/\gamma)^2]\,\{[D^2 - (\delta/\gamma)^2]^2 - (M/\gamma)^2 D^2\}\Psi = -(\delta^2 R/\gamma^6)\Psi, \quad (15)$$

with boundary conditions

$$\Psi(\pm 1) = D\Psi(\pm 1) = \{[D^2 - (\delta/\gamma)^2]^2 - (M/\gamma)^2 D^2\}\Psi(\pm 1) = 0. \quad (16)$$

Equation (15) and the boundary conditions (16) are identical with those found for the Bénard problem with vertically applied magnetic field [2] and hence have the same characteristic values. For $M = 0$, the lowest value of the Rayleigh number is

$$R^{1/4} = \left(\frac{1707{\cdot}8}{16}\right)^{1/4} \gamma = 3{\cdot}214\gamma, \quad (17)$$

and occurs at $\delta \simeq (\pi/2)\gamma$. It will be shown below that this relation (hereafter referred to as the Bénard limit) always yields a value lower than the true critical Rayleigh number at finite γ. When the magnetic interaction ($\sim M$) becomes large, equation (15) reduces to the inviscid form

$$D^2[D^2 - (\delta/\gamma)^2]\Psi = (\delta^2 R/\gamma^4 M^2)\Psi, \quad (18)$$

and is subject to the boundary conditions $\Psi(\pm 1) = D^2\Psi(\pm 1) = 0$. The characteristic relation corresponding to the lowest even mode of this equation is

$$R = \left(\frac{\pi}{2}\right)^2 (\gamma M)^2 \left[1 + \left(\frac{\pi\gamma}{2\delta}\right)^2\right], \quad (19)$$

from which it follows that the minimum value of R corresponds to disturbances of zero wave length ($\delta \to \infty$).

4. SOLUTION FOR ZERO ASPECT RATIO

The second limiting form of equations (9) and (10) is obtained by setting $\gamma = 0$. Physically this corresponds to the stability of a fluid between infinite vertical planes [8]. The equation governing stability is

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(\frac{\mathrm{d}^4}{\mathrm{d}\eta^4} - R\right)\psi = 0 \quad (20)$$

with boundary conditions $\psi = \mathrm{d}\psi/\mathrm{d}\eta = \mathrm{d}^4\psi/\mathrm{d}\eta^4 = 0$ at $\eta = \pm 1$. It will be noted that this equation is independent of the applied magnetic field. Solutions of (20) are identical with those of a vibrating beam clamped at both ends. They have the even and odd forms

$$\left.\begin{aligned}\psi_{\text{even}} &= \frac{\cosh \lambda\eta}{\cosh \lambda} - \frac{\cos \lambda\eta}{\cos \lambda},\\ \psi_{\text{odd}} &= \frac{\sinh \tau\eta}{\sinh \tau} - \frac{\sin \tau\eta}{\sin \tau},\end{aligned}\right\} \quad (21)$$

where $\tau, \lambda \equiv R^{1/4}$. The corresponding characteristic values are determined by the transcendental equations

$$\tanh \lambda + \tan \lambda = 0, \quad \coth \tau - \cot \tau = 0 \quad (22)$$

and have the values $\lambda = 2{\cdot}3650,\ 5{\cdot}4978, \ldots$ and $\tau = 3{\cdot}9266,\ 7{\cdot}0686, \ldots$. Convection will first occur when the temperature gradient exceeds the value $\beta \simeq 31{\cdot}3\, k\nu/g\alpha a^4$.

5. NUMERICAL SOLUTION FOR ARBITRARY γ

To solve the characteristic value equations for arbitrary gap aspect ratio, we assume that the dependent variables can be represented by the infinite series

$$\left.\begin{aligned}\psi &= \sum_{m,n=1}^{\infty} A_{mn} U_m(\eta) V_n(\zeta),\\ \theta &= \sum_{m,n=1}^{\infty} B_{mn} X_m(\eta) Y_n(\zeta),\end{aligned}\right\} \quad (23)$$

where A_{mn} and B_{mn} are expansion coefficients and $[U_m(\eta)V_n(\zeta)]$, $[X_m(\eta)Y_n(\zeta)]$ a set of functions satisfying the boundary conditions (11). Substituting (23) into equations (9) and (10), one obtains

$$\sum_{m,n=1}^{\infty} A_{mn}\left[L^2 - (\gamma M)^2 \frac{\partial^2}{\partial\zeta^2}\right] U_m V_n - R B_{mn} \frac{\partial X_m}{\partial\eta} Y_n = 0, \quad (24)$$

$$\sum_{m,n=1}^{\infty} B_{mn} L(X_m Y_n) - A_{mn} \frac{\partial U_m}{\partial\eta} V_n = 0. \quad (25)$$

The expansion coefficients in these equations can be evaluated by a modified Fourier technique [9]. Essentially the method consists of multiplying (24) by the function $U_k(\eta)V_l(\zeta)$ and (25) by the function $X_k(\eta)Y_l(\zeta)$, and then integrating over the range $-1 \leqslant \eta \leqslant 1$, $-1 \leqslant \zeta \leqslant 1$. This procedure leads to an infinite set of linear algebraic equations for the coefficients A_{mn} and B_{mn}. The requirement that these coefficients have non-trivial values, yields the infinite secular determinant

$$\begin{vmatrix} \Gamma^{11}_{11}(\gamma M) & \Gamma^{11}_{12}(\gamma M) & \cdot & -R\Gamma^{11}\Delta_{11} & -R\Gamma^{11}\Delta_{12} & \cdot \\ \Gamma^{11}_{12}(\gamma M) & \Gamma^{12}_{12}(\gamma M) & \cdot & -R\Gamma^{12}\Delta_{11} & -R\Gamma^{12}\Delta_{12} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline -\Delta^{11}\Gamma_{11} & -\Delta^{11}\Gamma_{12} & \cdot & {}^{11}_{11}(\gamma) & \Delta^{11}_{12}(\gamma) & \cdot \\ -\Delta^{12}\Gamma_{11} & -\Delta^{12}\Gamma_{12} & \cdot & \Delta^{11}_{12}(\gamma) & \Delta^{12}_{12}(\gamma) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0, \quad (26)$$

where

$$\Gamma^{kl}_{mn}(\gamma, M) = \int_{-1}^{+1}\int_{-1}^{+1} U_k V_l \left[L^2 - (\gamma M)^2 \frac{\partial^2}{\partial\zeta^2}\right] \times U_m V_n \,\mathrm{d}\eta\,\mathrm{d}\zeta,$$

$$\Gamma^{kl}\Delta_{mn} = \int_{-1}^{+1}\int_{-1}^{+1} U_k V_l \left(\frac{\partial}{\partial\eta} X_m\right) Y_n \,\mathrm{d}\eta\,\mathrm{d}\zeta,$$

$$\Delta^{kl}\Gamma_{mn} = \int_{-1}^{+1}\int_{-1}^{+1} X_k Y_l \left(\frac{\partial}{\partial\eta} U_m\right) V_n \, d\eta \, d\zeta,$$

$$\Delta^{kl}_{mn}(\gamma) = \int_{-1}^{+1}\int_{-1}^{+1} X_k Y_l \, L(X_m Y_n) \, d\eta \, d\zeta.$$

By choosing suitable expansion functions (i.e. functions resembling the actual modes of convection), it is possible to obtain good approximations for the lowest values of R by truncating determinant (26) at finite $k, l, m, n = N$. In the present analysis we chose to expand ψ in terms of the beam functions (21) and θ in terms of sine or cosine functions according to the symmetry suggested by equations (24) and (25). Specifically, the two physically relevant sets of expansion functions used were

$$\left.\begin{aligned} U_m(\eta)V_n(\zeta) &= [\cosh \lambda_m\eta + q_m \cos \lambda_m\eta]\,[\cosh \lambda_n\zeta + q_n \cos \lambda_n\zeta], \\ X_m(\eta)Y_n(\zeta) &= \sin\left(\frac{2m-1}{2}\right)\pi\eta \cos\left(\frac{2n-1}{2}\right)\pi\zeta, \end{aligned}\right\} \quad (27)$$

and

$$\left.\begin{aligned} U_m(\eta)V_n(\zeta) &= [\sinh \tau_m\eta + r_m \sin \tau_m\eta]\,[\cosh \lambda_n\zeta + q_n \cos \lambda_n\zeta], \\ X_m(\eta)Y_n(\zeta) &= \cos m\pi\eta \cos\left(\frac{2n-1}{2}\right)\pi\zeta, \end{aligned}\right\} \quad (28)$$

where $q_m = -\cosh \lambda_m/\cos \lambda_m$ and $r_m = -\sinh \tau_m/\sin \tau_m$. The expansions were terminated at $N = 3$ so that the evaluation of R involved the solution of an 18×18 determinant.

6. DISCUSION OF RESULTS

Results of the numerical analysis are summarized in Figs 1–3. The onset of convection is characterized by the appearance of horizontal rolls whose number increases both as γ and as M increase. At $M = 0$, the instability manifests itself in form of a single convective roll (1,1 mode) whenever $\gamma < 1{\cdot}6$ and as a double convective roll (2,1 mode) for $1{\cdot}6 < \gamma < 2{\cdot}6$. A comparison with the Bénard limit shows that the critical Rayleigh number associated with these modes is always in excess of the value predicted by (17). This difference is a direct

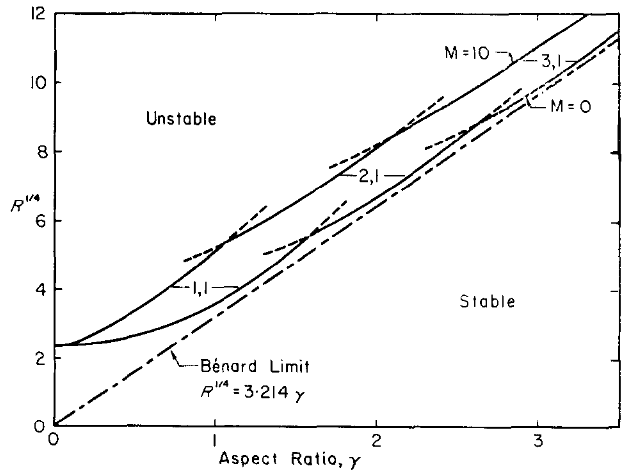

FIG. 1. Variation of Rayleigh number as a function of aspect ratio for $M = 0$ and $M = 10$.

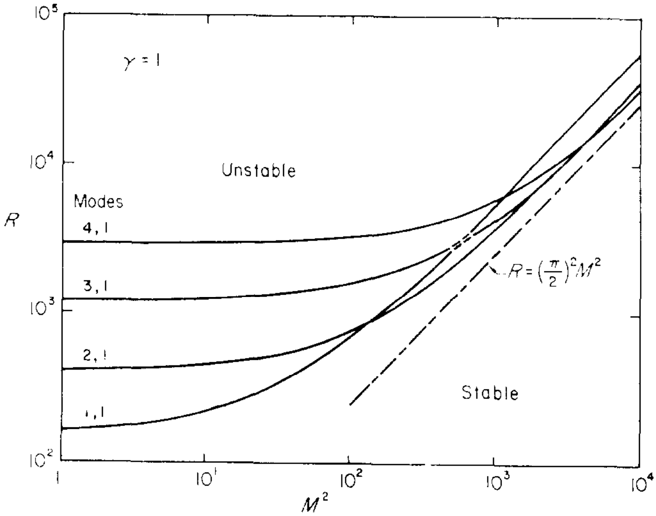

FIG. 2. Variation of Rayleigh number as a function of Hartmann number for a cavity of square cross section.

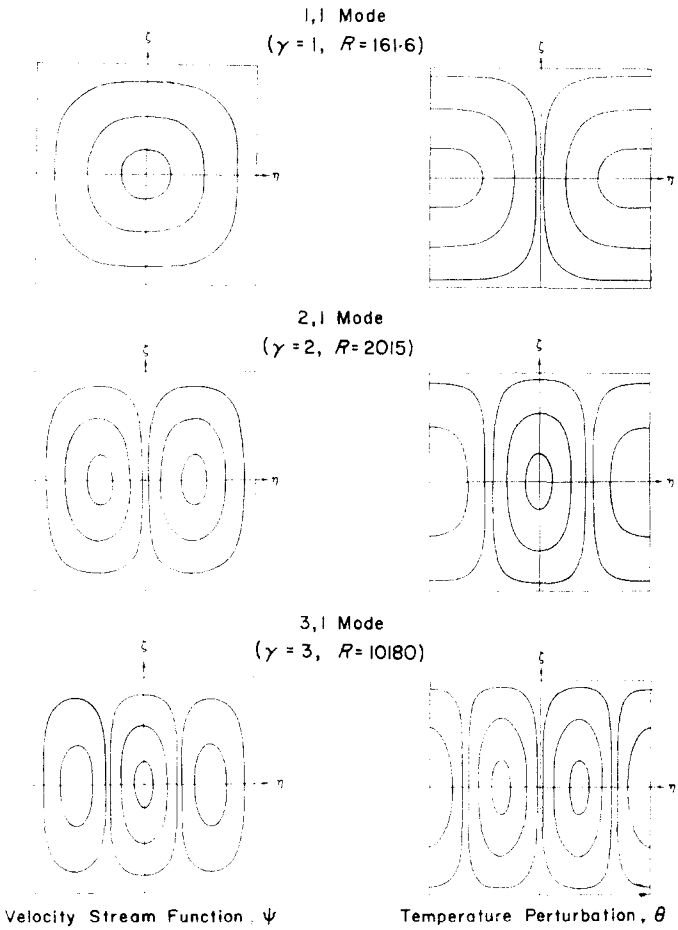

FIG. 3. Typical modes of convection at the onset of instability in the absence of a magnetic field. The contours represent the iso-values $\psi/\psi_{\max}$, $\theta/\theta_{\max} = \pm 0{\cdot}9$, $\pm 0{\cdot}5$, $\pm 0{\cdot}1$.

consequence of viscous damping due to the side-walls and becomes appreciable as $\gamma \to 0$. At $\gamma = 1$, $M = 0$, we find $R = 161{\cdot}6$, a value about half as large as that obtained by Velte [6] for a cavity with perfectly conducting walls. The effect of the magnetic field is to hinder the onset of convection and to increase the number of horizontal rolls occurring for a given aspect ratio. For very large fields, the critical Rayleigh number is proportional to the square of the Hartmann number, in accordance with equation (19). Typical modes at the onset of convection for zero magnetic field, calculated by evaluating the expansion coefficients A_{mn} and B_{mn} for fixed R and γ, are shown in Fig. 3. Application of a magnetic field does not appreciably alter the shape of these convection cells, but rather causes a transition to a higher mode.

ACKNOWLEDGEMENT

I am indebted to Mr. J. Boris for valuable assistance with the numerical work.

REFERENCES

1. W. B. Thompson, Thermal convection in a magnetic field, *Phil. Mag.* **42**, 1417–1432 (1951).
2. S. Chandrasekhar, On the inhibition of convection by a magnetic field, *Phil. Mag.* **43**, 501–532 (1952).
3. Y. Nakagawa, Experiments on the inhibition of thermal convection by a magnetic field, *Proc. Roy. Soc.* A **240**, 108–113 (1957).
4. S. A. Regirer, Stationary convective motion of a conducting fluid between parallel vertical plates in a magnetic field, *Sov. Phys., J. Eksp. Teor. Phys.* **10**, 149–152 (1960).
5. V. S. Sorokin and I. V. Sushkin, Stability of equilibrium of a conducting liquid heated from below in a magnetic field, *Sov. Phys., J. Eksp. Teor. Phys.* **11**, 440–445 (1960).
6. W. Velte, Zur Stabilität der Strömung in einem horizontalen Rohr bei ungleichmässig erwärmter Wand, *Z. angew. Math. Phys.* **13**, 591–600 (1962).
7. J. M. Mihahjan, A rigorous exposition of the Boussinesq approximation applicable to a thin layer of fluid, *Astrophys. J.* **136**, 1126–1133 (1962).
8. C. S. Yih, Thermal instability of viscous fluids, *Quart. Appl. Math.* **17**, 25–42 (1959).
9. G. Poots, Heat transfer by laminar free convection in enclosed plane gas layers, *Quart. Mech. Appl. Math.* **11**, 257–273 (1958).

Résumé—Le démarrage de la convection dans un fluide conducteur de l'électricité contenu dans une cavité infinie de section droite rectangulaire est examiné dans le cas où le fluide est soumis à la fois à un gradient de température vertical et à un champ magnétique vertical. Les équations aux valeurs caractéristiques gouvernant la stabilité du fluide sont obtenues et une technique de Fourier modifiée est employée pour les résoudre. On trouve que le gradient critique de température pour le démarrage de la convection est une fonction du nombre de Hartmann et de l'allongement de la cavité.

Zusammenfassung—Der Beginn der Konvektion in einer elektrisch leitenden Flüssigkeit, die auf eine unbegrenzte Vertiefung von rechteckigem Querschnitt beschränkt ist, wird für den Fall, dass die Flüssigkeit sowohl einem vertikalen Temperaturgradienten wir auch einem Magnetfeld unterliegt, untersucht. Gleichungen charakteristischer Werte für die Regelung der Flüssigkeitsstabilität werden abgeleitet und eine abgeänderte Fouriertechnik zu ihrer Lösung verwendet. Der kritische Temperaturgradient für den Konvektionsbeginn ergibt sich als Funktion der Hartmannzahl und des Verhältnisses von Breite zu Höhe der Vertiefung.

Аннотация—Исследовалось начало конвекции в электрически проводящей жидкости, заключенной в неограниченную полость четырехугольного сечения, для случая, когда жидкость находится под воздействием как вертикального температурного градиента, так и вертикального магнитного поля. Выведены характеристические уравнения, описывающие устойчивость жидкости, и для их решения используется несколько видоизмененный метод Фурье. Найдено, что критический температурный градиент, при котором начинается конвекция, является функцией критерия Хартмана и относительного удлинения полости.